UDC 531.36

ON THE STABILITY OF PERIODIC MOTIONS UNDER RESONANCE

PMM Vol. 39, №1, 1975, pp. 36-44

A. L. KUNITSYN
(Moscow)

(Received June 11, 1974)

We investigate the stability of the zero solution of a nonlinear system of differential equations with periodic coefficients and holomorphic right-hand sides. We examine the critical case when the characteristic equation of the linearized system has only complex conjugate roots equal to unity in absolute value, while specific integer relations (resonance) exist between the characteristic indices and the frequency of the unperturbed motion. We formulate the conditions under which the stability question is resolved by the first nonlinear forms of the expansion of the right-hand sides of the equations of perturbed motion. For the most important types of resonance we have obtained the necessary and sufficient conditions for stability with respect to even-order forms. The sufficiency is proved by the existence of a sign-definite integral, while the necessity, by the construction of a Chetaev function. The results obtained are extended, in particular, to the stability of the periodic motions of Hamiltonian systems. Special cases of this problem are examined in [1, 3]. A general approach to solving this problem for non-Hamiltonian systems of second order was developed in [4].

1. We consider the system of equations of perturbed motion

$$dx_* / dt = X_* (x_*, t) \tag{1.1}$$

$$X_* (x_*, t) = A(t) x_* + \sum_{l=m\geqslant 2}^{\infty} X_*^{(l)} (x_*, t)$$

where x_* is a $2n$-dimensional vector and $X_*\ (x_*, t)$ is an analytic vector-valued function, periodic in t with a real period ω, of the form indicated.

Let the matrix $A\ (t)$ be such that all the roots of the characteristic equation are complex and equal to unity in absolute value. Then, the problem of the stability [5] of the trivial solution of the nonautonomuous system (1.1) reduces to the critical case of the stability of n pairs of pure imaginary roots for the autonomous system if between the characteristic indices $\pm \lambda_s\ (\lambda_s^2 < 0,\ s = 1, \ldots, n)$ and the number $2\pi i/\omega$ there exist no integer relations of the form

$$\langle P\Lambda\rangle = \frac{2\pi i}{\omega} p, \quad i = \sqrt{-1} \quad p = 0, \pm 1, \pm 2, \ldots \tag{1.2}$$

$$P = (p_1, \ldots, p_n), \quad |P| = p_1 + \ldots + p_n \geqslant 3, \quad p_s \geqslant 0$$

$$\Lambda = (\lambda_1, \ldots, \lambda_\alpha, \lambda_{\alpha+1}, \ldots, \lambda_n)$$

$$\lambda_j i > 0, \quad j = 1, \ldots, \alpha, \quad \lambda_s i < 0, \quad s = \alpha + 1, \ldots, n$$

where P is an n-dimensional vector with integer components.

We investigate the problem of the stability of the trivial solution of system (1.1) when the resonance relations (1.2) are satisfied. Here we consider the case when the stability question is resolved by m-th-order forms, representing the lowest nonlinear terms in the expansion of the vector-valued function $X_*(x_*, t)$. As will be clear from what follows, in this case we must have $|P| = m + 1$. We shall examine only even values of m and, in addition, assume that vector P, satisfying (1.2), is unique. By the same token we have excluded integer and half-integer values of λ_s as well as the cases of complex resonance when the same frequencies are encountered in different resonance relations.

Using a linear transformation without violating the stability problem, system (1.1) can be written as [5]

$$x^{\cdot} = \lambda x + \sum_{l=m\geqslant 2}^{\infty} X^{(l)}(x, y, t), \quad y^{\cdot} = -\lambda y + \sum_{l=m\geqslant 2}^{\infty} Y^{(l)}(x, y, t) \qquad (1.3)$$

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad \lambda = \{\lambda_1, \ldots, \lambda_n\}$$

Here x, y are complex-conjugate vectors, λ is a diagonal matrix, $X^{(l)}(x, y, t)$ and $Y^{(l)}(x, y, t)$ are complex-conjugate vector-valued functions, periodic in t with period ω, whose components $X_s^{(l)}$ and $Y_s^{(l)}$ are represented by lth-order forms, so that

$$X_s^{(l)} = \sum_{|k_s|+|l_s|=l} R_{k_s,l_s}(t)\, x_1^{k_{s1}} \ldots x_n^{k_{sn}} y_1^{l_{s1}} \ldots y_n^{l_{sn}}$$

$$R_{k_s,l_s}(t) \equiv R_{k_s,l_s}(t+\omega), \quad s = 1, 2, \ldots, n$$

$$k_s = (k_{s1}, \ldots, k_{sn}), \quad l_s = (l_{s1}, \ldots, l_{sn}), \quad k_{sj}, l_{sj} \geqslant 0$$

$$|k_s| = k_{s1} + \ldots + k_{sn}, \quad |l_s| = l_{s1} + \ldots + l_{sn}$$

where k_s, l_s are all possible integer vectors. We transform (1.3) by the replacement

$$x_s = [u_s - \sum_m U_{k_s,l_s}(t)\, u_1^{k_s} \ldots u_n^{k_{sn}} v_1^{l_{s1}} \ldots v_n^{l_{sn}}] \exp(\lambda_s t) \qquad (1.4)$$

$$y_s = [v_s - \sum_m V_{k_s,l_s}(t)\, v_1^{k_{s1}} \ldots v_n^{k_{sn}} u^{l_{s1}} \ldots u_n^{l_{sn}}] \exp(-\lambda_s t)$$

$$s = 1, 2, \ldots, n$$

in which we try to select the complex-conjugate functions $U_{k_s,l_s}(t)$ and $V_{k_s,l_s}(t)$ such that they are bounded in t, while in the transformed equations the mth-order forms have constant coefficients. The first group of equations in the complex-conjugate variables u_s and v_s takes the form

$$u_s^{\cdot} = \sum_m [U^{\cdot}_{k_s,l_s} + R_{k_s,l_s}(t) \exp(\varkappa_s t)]\, u_1^{k_{s1}} \ldots u_n^{k_{sn}} v_1^{l_{s1}} \ldots v_n^{l_{sn}} + \ldots \qquad (1.5)$$

$$\varkappa_s = \sum_{j=1}^{n} (k_{sj} - l_{sj}) \lambda_j - \lambda_s, \quad s = 1, 2, \ldots, n$$

where the unwritten terms have an order of smallness no lower than $m + 1$.

We select the functions $U_{k_s,l_s}(t)$ from the equations

$$U^{\cdot}_{k_s,l_s} + R_{k_s,l_s}(t) \exp(\varkappa_s t) = g_{k_s,l_s} \qquad (1.6)$$

where g_{k_s,l_s} are certain constants which we determine from the requirement of boundedness of $U_{k_s,l_s}(t)$. Expanding $R_{k_s,l_s}(t)$ in a Fourier series, we have

$$R_{k_s,l_s}(t)\exp(\varkappa_s t) = \sum_{-\infty}^{+\infty} b_{k_s,l_s}^{(n)} \exp\left(\frac{2\pi n i}{\omega} + \varkappa_s\right) t$$

To obtain bounded U_{k_s,l_s} when $\varkappa_s$ does not satisfy the condition

$$\varkappa_s = \frac{2\pi i}{\omega} p, \qquad p = 0, \pm 1, \pm 2, \ldots \tag{1.7}$$

it is obviously necessary to set $g_{k_s,l_s} = 0$. The $U_{k_s,l_s}(t)$ themselves are represented in the series form

$$U_{k_s,l_s}(t) = \sum_{-\infty}^{+\infty} \frac{b_{k_s,l_s}^{(n)}}{\varkappa_s + 2\pi n i/\omega} \exp\left[\left(\varkappa_s + \frac{2\pi n i}{\omega}\right) t\right] + C_{k_s,l_s}$$

which, obviously, converge since the quantity $\varkappa_s$ remains unchanged for each pair of vectors (k_s, l_s). Thus, we can suppress all the nonresonant terms in the mth-order forms.

Now let (1.7) be satisfied for some $p = p_*$. Then, setting $g_{k_s,l_s} = b_{k_s,l_s}^{(p*)}$, from (1.6) we obtain

$$U_{k_s,l_s}(t) = \sum_{-\infty}^{+\infty} \frac{b_{k_s,l_s}^{(n)}}{(n + p_*) 2\pi n i/\omega} \exp\left[(n + p_*) \frac{2\pi n i}{\omega} t\right] + C_{k_s,l_s}$$

where $n + p_* \neq 0$ and, consequently, in this case all the U_{k_s,l_s} are periodic with period ω. Comparing (1.2) and (1.7) we can be convinced that under the assumptions made concerning the vectors P and Λ the relations (1.2) are satisfied by the unique vectors k_s and l_s, namely:

for $s = 1, \ldots, \alpha$: $k_{sj} = 0, \quad l_{sj} = p_j, \quad l_{ss} = p_s - 1, \quad j = 1, \ldots, \alpha$

$\qquad k_{sj} = p_j, \quad {}_{sj} = 0, \quad j = \alpha + 1, \ldots, n$

for $s = \alpha + 1, \ldots, n$; $\quad k_{sj} = p_j, \quad l_{sj} = 0, \quad j = 1, \ldots, \alpha$

$\qquad k_{sj} = 0, \quad l_{sj} = p_j, \quad l_{ss} = p_s - 1, \quad j = \alpha + 1, \ldots, n$

The constant coefficients g_{k_s,l_s} corresponding to the vectors k_s and l_s indicated, are

$$g_{k_s,l_s} = b_{k_s,l_s}^{(p_*)} = \frac{1}{\omega} \int_0^{\omega} R_{k_s,l_s}(t) \exp\left(-\frac{2\pi p i}{\omega} t\right) dt$$

and since a unique pair of resonance vectors k_s, l_s exists for each $s = 1, \ldots, n$, in the nonsingular case $(q_{k_s,l_s} \neq 0)$ there is only one nonzero complex coefficient g_{k_s,l_s} in each of the equations.

Setting

$$g_{k_s,l_s} = a_s - ib_s, \quad s = 1, \ldots, \alpha, \qquad g_{k_s,l_s} = a_s + ib_s, \quad s = \alpha + 1, \ldots, n$$

we reduce the first group of complex-conjugate equations to the form

$$v_s u_s^{\cdot} = (a_s - ib_s) v_1^{p_1} \ldots v_\alpha^{p_\alpha} u_{\alpha+1}^{p_{\alpha+1}} \ldots u_n^{p_n} + \Phi_s(u, v, t), \; s = 1, \ldots, \alpha \tag{1.8}$$

$$v_s u_s^{\cdot} = (a_s + ib_s) u_1^{p_1} \ldots u_\alpha^{p_\alpha} v_{\alpha+1}^{p_{\alpha+1}} \ldots v_n^{p_n} + \Phi_s(u, v, t), \; s = \alpha + 1, \ldots, n$$

where the functions $\Phi_s(u, v, t)$ are bounded in the region

$$\sum_{s=1}^{n} u_s v_s \leqslant h, \quad h > 0, \quad t \in [t_0, \infty]$$

and are of orders not less than $m + 2$ in u_s and v_s.

Now setting

$$u_s = r_s \exp(-i\theta_s), \quad v_s = r_s \exp(i\theta_s), \qquad s = \alpha + 1, \ldots, n$$

$$u_s = r_s \exp(i\theta_s), \quad v_s = r_s \exp(-i\theta_s), \qquad s = 1, \ldots, n$$

and adding onto Eqs. (1.8) the unwritten group of equations conjugate to them, we reduce the complete system of equations of perturbed motion to the following form, independently of the number $\alpha = 1, \ldots, n$ characterizing the resonance vectors P and Λ:

$$r_s r_s^{\cdot} = Q_s(\theta) \prod_{j=1}^{n} r_j^{p_j} + R_s(r, \theta, t) \tag{1.9}$$

$$s = 1, \ldots, n$$

$$r_s^2 \theta_s^{\cdot} = \frac{\partial Q_s}{\partial \theta} \prod_{j=1}^{n} r_j^{p_j} + \Theta_s(r, \theta, t)$$

Here

$$Q_s(\theta) = a_s \cos\theta + b_s \sin\theta$$

$$\theta = p_1\theta_1 + \ldots + p_n\theta_n, \quad r = (r_1^2 + \ldots + r_n^2)^{1/2}$$

while the functions $R_s(r, \theta, t)$ and $\Theta_s(r, \theta, t)$, being 2π-periodic in θ and almost periodic in t, are of an order of smallness not less than $m + 2$ relative to r. The system of equations resulting from (1.9) when $R_s = \Theta_s \equiv 0$, is subsequently called the model system.

2. We reduce the investigation of the stability of the trivial solution of system (1.9) to three principally different cases

$$n = 1, \quad (m+1)\lambda = \frac{2\pi i}{\omega} p \tag{2.1}$$

$$n = 2, \quad p_1\lambda_1 \pm p_2\lambda_2 = \frac{2\pi i}{\omega} p, \quad p_1 + p_2 = m + 1$$

$$n = 3, \quad p_1\lambda_1 + p_2\lambda_2 \pm p_3\lambda_3 = \frac{2\pi i}{\omega} p, \quad p_1 + p_2 + p_3 = m + 1$$

where $p = 0, \pm 1, \pm 2, \ldots$, while $p_s > 0$ are integers.

Let us consider the case $n = 1$. System (1.9) can be reduced to the form

$$rr^{\cdot} = \sqrt{a^2 + b^2}\, r^{m+1} \cos[\psi - (m+1)\theta] + \ldots \tag{2.2}$$

$$r^2\theta^{\cdot} = \sqrt{a^2 + b^2}\, r^{m+1} \sin[\psi - (m+1)\theta] + \ldots$$

$$\sin\psi = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos\psi = \frac{a}{\sqrt{a^2 + b^2}}$$

The model system has $2(m + 1)$ singular directions, determined by the equations

$$\theta_q = (\psi + q\pi)/(m+1), \quad q = 1, 2, \ldots, 2(m+1)$$

It can be verified that one-half of these directions correspond to unstable particular solutions (an unstable ray), while the equilibrium position is a singular saddle point.

Theorem 2.1. The trivial solution of system (2.2) is unstable for $a^2 + b^2 \neq 0$.

The proof can be carried out with the aid of the Liapunov function

$$V = r^{m+1} \cos[\psi - (m+1)\theta]$$

whose derivative by virtue of (2.2) is

$$V^{\cdot} = (m+1)\sqrt{a^2+b^2}r^{2m} + r^{2m+1}\Phi(r, \theta, t)$$

where $\Phi(r, \theta, t)$ is a function 2π-periodic in θ and almost periodic in t, consequently, is bounded in every region $r < h$. Obviously, for

$$\psi - {}^1/_2\pi < (m+1)\theta < \psi + {}^3/_2\pi$$

we have $VV^{\cdot} > 0$, which satisfies Liapunov instability theorem [6] since $V^{\cdot}$ is a positive-definite function in a whole neighborhood of the origin for a fairly small r. We note that a special case of the one being considered was investigated in [1-5] wherein only Hamiltonian systems were studied.

Let us investigate the influence of this resonance in a system of arbitrary order, having $n+k$ pairs of pure imaginary characteristic indices, of which the first n pairs do not satisfy even one of relations (1.2), where again $|P| = m+1$, while m (even) is the degree of the lowest of the nonlinear forms with which the expansions of the right-hand sides of the original differential equations start. The remaining k pairs of characteristic indices satisfy only resonance relations of the form

$$\lambda_j^*(m+1) = \frac{2\pi i}{\omega}p, \quad p = 0, \pm 1, \pm 2, \ldots, \quad j = 1, 2, \ldots, k \tag{2.3}$$

Then the original system of equations can be represented as

$$\begin{aligned} &x^{\cdot} = \lambda x + X(x, y, \xi, \eta, t), \quad \xi^{\cdot} = \lambda^*\xi + \Xi(x, y, \xi, \eta, t) \\ &y^{\cdot} = -\lambda y + Y(x, y, \xi, \eta, t), \quad \eta^{\cdot} = -\lambda^*\eta + H(x, y, \xi, \eta, t) \\ &x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad \xi = (\xi_1, \ldots, \xi_k) \\ &\eta = (\eta_1, \ldots, \eta_k), \quad \lambda = \{\lambda_1, \ldots, \lambda_n\}, \quad \lambda^* = \{\lambda_1^*, \ldots, \lambda_k^*\} \end{aligned} \tag{2.4}$$

Here X, Y and Ξ, H are complex-conjugate vector-valued functions, periodic in t, whose expansions with respect to x, y, ξ, η start with mth-order terms.

Using the nonlinear transformation (1.4), system (2.4) can be brought to the form

$$\begin{aligned} &\rho_s\rho_s^{\cdot} = \Psi_s, \quad \rho_s^2\varphi_s^{\cdot} = \Phi_s \\ &r_j r_j^{\cdot} = \sqrt{a_j^2+b_j^2}\,r^{m+1}\cos[\psi_j - (m+1)\theta_j] + R_j \\ &r_j^2\theta_j^{\cdot} = \sqrt{a_j^2+b_j^2}\,r^{m+1}\sin[\psi_j - (m+1)\theta_j] + \Theta_j \\ &r = (r_1^2 + \ldots + r_k^2)^{1/2}, \ \rho = (\rho_1^2 + \ldots + \rho_n^2)^{1/2}, \ j = 1, \ldots, k, \ s = 1, \ldots, n \end{aligned} \tag{2.5}$$

where the functions Ψ_s, Φ_s, R_j, Θ_j are bounded relative to $\varphi_1, \ldots, \varphi_n, \theta_1, \ldots, \theta_k$ and t, while their expansions with respect to r and ρ start with terms of order not less than $m+2$. The following theorem proves to be valid for the system obtained.

Theorem 2.2. If even one of the inequalities $a_j^2 + b_j^2 \neq 0$, $j = 1, \ldots, k$, is fulfilled, the zero solution of system (2.5) and, consequently, (2.4), is unstable.

The proof can be given with the aid of Kamenkov instability theorem [7] in the case of $n+k$ zero roots of the charactersitc equation.

Let $X_j^{(m)}$ and $Y_j^{(m)}$ be the mth-order forms with which the expansions of the vector-valued functions X and Y of system (2.4) start. We set up the forms

$$F_j = x_j Y_j^{(m)} - y_j X_j^{(m)} \equiv \sqrt{a_j^2+b_j^2}\,r^{m+1}\sin[\psi - (m+1)\theta_j]$$

$$j = 1, 2, \ldots, k$$

$$G_j = x_j X_j^{(m)} + y_j Y_j^{(m)} \equiv \sqrt{a_j^2+b_j^2}\,r^{m+1}\cos[\psi - (m+1)\theta_j]$$

According to Kamenkov theorem the trivial solution of system (2.4) is unstable independently of the terms of higher than mth-order if for even one value θ_j^*, being a root of the equation $F_j(\theta_j) = 0$, the form $G_j(\theta_j^*) > 0$. Suppose that we have $a_j^2 + b_j^2 \neq 0$ for even one pair a_j, b_j. Then, by setting $\theta_j^* = (m+1)^{-1}\psi$, we obviously satisfy all the hypotheses of Kamenkov theorem.

The theorem we have proved allows us to draw the following conclusion: the presence of even one resonance of form (2.3) in a system leads, as a rule (excepting total degeneracy), to the instability of the whole system. In the case $m = 2$, of practical importance, when designing the system we should avoid the situation when among the characteristic indices there is even one which is an integral multiple of one-third of the frequency of the unperturbed periodic motion.

We now consider the cases $n = 2$, $n = 3$ from (2.1). In both these cases the model system resulting from (1.9) by discarding the functions R_s and Θ_s has one and the same form as in the case of odd-order integral resonance for autonomous systems, treated in fullest generality in [8, 9]. Therefore, all the conclusions on the stability drawn in these papers are preserved here. We cite them without proof.

Theorem 2.3. Under resonance in the case $n = 2$ the necessary and sufficient condition for the stability of the model system is

$$a_1 / a_2 = b_1 / b_2 < 0$$

The sufficiency can be proved by the existence of a sign-definite integral $a_2 r_1^2 - a_1 r_2^2 = \text{const}$, while the necessity, by Chetaev theorem, where the Chetaev function we can take in the form

$$V = r_1^{p_1} r_2^{p_2} (\cos\theta + \varkappa \sin\theta)$$

Theorem 2.4. Under resonance in the case $n = 3$ the necessary and sufficient conditions for the stability of the model system are

$$\beta_j \beta_s > 0, \qquad j, s = 1, 2, 3$$

$$\beta_1 = a_2 b_3 - a_3 b_2, \quad \beta_2 = a_3 b_1 - a_1 b_3, \quad \beta_3 = a_1 b_2 - a_2 b_1$$

The sufficiency again can be proved by the sign-definite integral $\beta_1 r_1^2 + \beta_2 r_2^2 + \beta_3 r_3^2 = \text{const}$, while the necessity, by the Chetaev function

$$V = r_1^2 r_2^2 r_3^2 (\cos\theta + \varkappa \sin\theta)$$

In the three cases of resonance considered a tendency is revealed of a decrease in the cases of instability with an increase in the number of frequencies participating in the resonance: in the case $n = 1$ stability is possible only under total degeneracy $(a^2 + b^2 = 0)$, while for $n = 2$ one case of stability is possible, and for $n = 3$ the stability of the model system is preserved in the whole region of values of its parameters.

Dwelling on the question of the connection of the stability problem for the complete system with the stability problem for the model system, we note that from the stability of the model system it is impossible to draw any conclusions on the stability of the complete system, while the instability of the model system necessarily implies the instability of the complete system. This fact has been rigorously proved (*) for autonomous systems.

*) Nurpeisov, S., On stability in the critical case of n pairs of pure imaginary roots in the presence of internal resonance. Candidate's Dissertation. Alma-Ata, 1972.

As a consequence of the boundedness with respect to time of the forms of higher than mth order in (1.8) and of the complete coincidence of the model systems for the autonomous case and for the case of periodic motions, the results on total instability of the autonomous system extend to the resonance case considered here for $n = 2$ and $n = 3$.

3. In conclusion we consider the Hamiltonian systems

$$\dot{x}_s^* = \frac{\partial H^*}{\partial y_s^*}, \quad \dot{y}_s^* = -\frac{\partial H^*}{\partial x_s^*}, \quad s = 1, 2, \ldots, n$$

where H^*, analytic in x_s^*, y_s^* and ω is the Hamiltonian function periodic in time t Then under the assumptions made above relative to the roots of the characteristic equation, with the aid of a linear transformation, ω-periodic in t, of the variables x_s^*, y_s^* to the new x_s, y_s we again obtain the canonic equations

$$x_s^{\cdot} = \frac{\partial H}{\partial y_s}, \quad y_s^{\cdot} = -\frac{\partial H}{\partial x_s}, \quad H = \sum_{s=1}^{n} \lambda_s x_s y_s + \sum_{l=m+1}^{\infty} H^{(l)}(x, y, t)$$

where x_s and y_s are complex-conjugate variables, while $H^{(l)}$ are lth-order forms in x_s, y_s with periodic coefficients. As we can easily convince ourselves, a canonic transformation of the equations obtained, analogous to (1.4), can be given by means of the generating function

$$S = \sum_{s=1}^{n} \Big[x_s v_s + \sum_{|k_s|+|l_s|=m+1} \Phi_{k_s, l_s}(t) x_1^{k_{s1}} \ldots x_n^{k_{sn}} v_1^{l_{s1}} \ldots v_n^{l_{sn}} \Big] \exp(-\lambda_s t)$$

Consequently, Eqs. (1.8) are again in canonic form if the original equations were canonic. If we require Eqs. (1.8) to have the form

$$u_s^{\cdot} = \partial K / \partial v_s, \quad v_s^{\cdot} = -\partial K / \partial u_s, \qquad s = 1, 2, \ldots, n$$

then under resonance in the case $n = 2$ we arrive at the condition $a_1 b_2 - a_2 b_1 = 0$, while for $n = 3$, to the condition $\beta_s = 0$, $s = 1, 2, 3$. It is interesting that for $n = 1$, for arbitrary original system, the model system corresponding to it is always obtained as canonic with the Hamiltonian

$$K = \frac{1}{m+1} [(a + ib) u^{m+1} + (a - ib) v^{m+1}]$$

Thus in the cases $n = 1$, $n = 2$ the stability conditions for the canonic systems follow in a special manner from Theorems 2.1 and 2.3.

The case $n = 3$ for canonic systems requires a special analysis. For autonomous canonic systems this case of resonance was investigated in [9] and, next, in [10] where, in particular, it was shown that the instability of the model system necessarily implies the instability of the complete system. In this case the following theorem gives the necessary and sufficient stability conditions for the model system.

Theorem 3.1. The necessary and sufficient condition for the stability of the trivial solution of the model system under resonance in the case $n = 3$ is the presence of an alternation of sign in the sequence of numbers a_1, a_2, a_3 or b_1, b_2, b_3.

The sufficiency can be proved by the sign-definite integral

$$\alpha_1 r_1^2 + \alpha_2 r_2^2 + \alpha_3 r_2^3 = \text{const}, \qquad \alpha_j > 0, \qquad j = 1, 2, 3$$

while the necessity, by the Chetaev function

$$V = r_1^{p_1} r_2^{p_2} r_3^{p_3} \cos \theta$$

The author thanks the participants of the seminar guided by V.V.Rumiantsev for discussing the paper.

REFERENCES

1. Malkin, I.G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
2. Levi-Cività, T., Sopra alcuni criteri da instabilità. Ann. di Mat. Pura et Appl., Ser. 3, Vol. 5, 1901.
3. Mustakhishev, K.M., On the stability of Hamiltonian systems. Trudy Univ. Druzhby Narodov, Vol. 27, №5, 1968.
4. Markeev, A.P., On the stability of a nonautonomous Hamiltonian system with two degrees of freedom. PMM Vol. 33, №3, 1969.
5. Kamenkov, G.V., Selected Works, Vol. 2, Moscow, "Nauka", 1971.
6. Liapunov, A.M., General Problem of the Stability of Motion. Collected Works, Vol. 2, Moscow, Izd. Akad. Nauk SSSR, 1956.
7. Kamenkov, G.V., Selected Works, Vol. 1, Moscow, "Nauka", 1971.
8. Kunitsyn, A.L., On stability in the critical case of pure imaginary roots under internal resonance. Differents. Uravneniia, Vol. 7, №9, 1971.
9. Kunitsyn, A.L., Stability in the critical case of three pairs of pure imaginary roots in the presence of internal resonance. PMM Vol. 35, №1, 1971.
10. Khazin, L.G., On the stability of Hamiltonian systems in the presence of resonances. PMM Vol. 35, №3, 1971.

Translated by N.H.C.

UDC 531.3

ON THE STABILITY OF NONLINEAR SYSTEMS OF THE NEUTRAL TYPE

PMM Vol. 39, №1, 1975, pp. 45-52
V.P.SKRIPNIK
(Moscow)
(Received May 3, 1973)

We prove that from the stability (asymptotic stability) of linear system (1) follows the stability (respectively, asymptotic stability) of the trivial solution of nonlinear system (2) if the deviations of the arguments and the nonlinear addition are sufficiently small in the correspinding integral sense.

For $l = 1,2,\ldots, q$ we denote $f(t, \xi_l, \eta_l) = f(t, \xi_1, \xi_2, \ldots, \xi_q, \eta_1, \eta_2, \ldots, \eta_q)$, where f, ξ_l, η_l are m-dimensional vectors. We consider the following two systems: the linear system (1) and the nonlinear system (2) perturbed [1] relative to (1)

$$y' = A(t)\,y, \qquad A(t) = \sum_{k=1}^{p} A_k(t) \tag{1}$$

$$x'(t) = \sum_{k=1}^{p} A_k(t)\,x(\varphi_k(t)) + f(t, x'(\psi_l(t)), x(\chi_l(t))) \tag{2}$$

Here φ_k, ψ_l, χ_l are transformations of the argument, $A_k(t)$ are square matrices, x and y are mth-order vectors. Everywhere the integrals are to be understood in the Lebesgue sense. The derivative is to be understood in the following sense. If for some